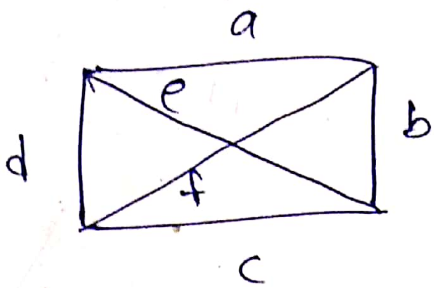
Planar Graphs

- A gph G is said to be planar if $\exists$ some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. otherwise called nonplanar.

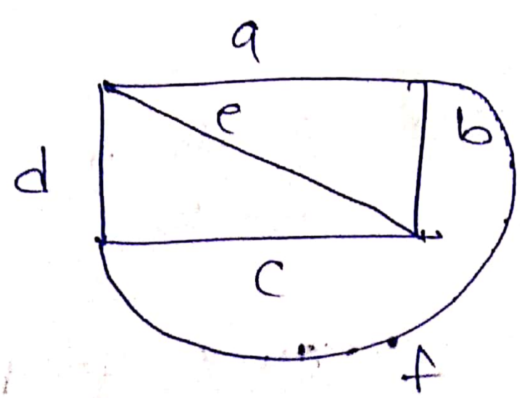
- A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

- Thus to ~~[illegible]~~ declare that a graph G is nonplanar, we have to show that of all possible geometric representation of G, none can be embedded in a plane. e.g.

G→



nonplanar but can be made planar



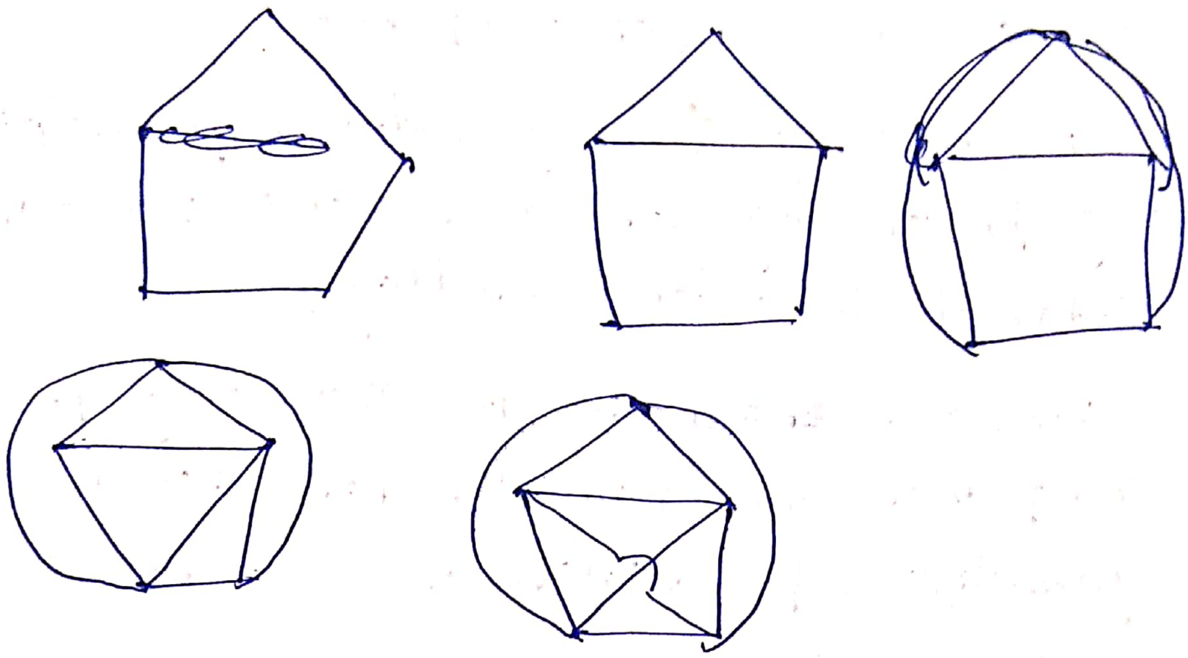
planar

$\therefore$ G is planar i.e. we have embedded the new geometric graph in the plane.

Kuratowski's Two Graphs →

First Graph → K_5 → complete graph with 5 vertices

Thm → The complete graph of five vertices is non-planar

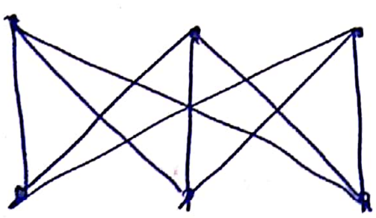


— Kuratowski's First graph → K_5

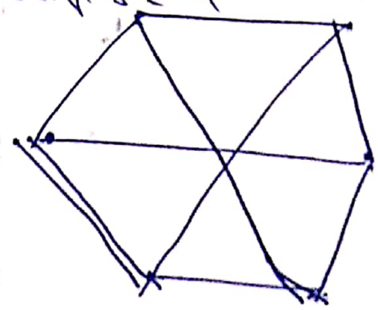
Second Graph → $K_{3,3}$ → complete bipartite graph

no. of vertices = 6

no. of edges = 9



or



Thm → K's second graph is also non-planar — $K_{3,3}$

Properties →

1. Both are regular graphs → all vertices are of equal degree
2. Both are non planar
3. Removal of one edge or a vertex makes each a planar graph

4. K_5 is the non planar graph with the smallest no. of vertices.

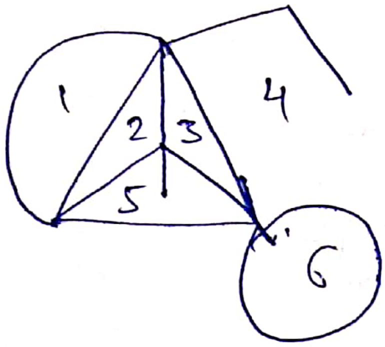
$K_{3,3}$ is the non planar graph with the smallest no. of edges.

Thus both are the simplest non planar graphs.

$K_{3,3}$ is also called utility graph.

Note → Any simple planar graph can be embedded in a plane s.t. every edge is drawn as a st. line segment.

Region → A plane representation of a graph divides the plane into regions (also called windows, faces, meshes) e.g.

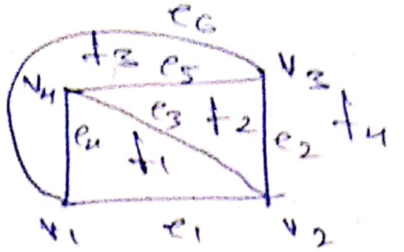


→ 6 regions, region 4 is infinite

- A region is characterized by the set of edges (or the set of vertices) forming its bdry.
- A region is finite if the area it encloses is finite otherwise it is infinite.

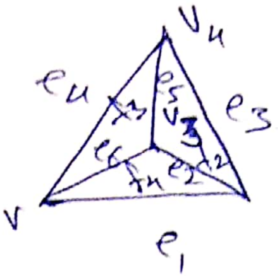
Note → A planar graph may be embedded in a plane st. any specified region can be made the infinite region. e.g.

e.g.



f_4 infinite (20)
→ characterized by
$\{e_1, e_2, e_6\}$

If we wish to make f_1 cha. by $\{e_1, e_3, e_4\}$ infinite then redraw the graph as below—



Euler's Formula → A connected graph with n vertices and e edges has $(e-n+2)$ regions i.e.
$\boxed{f = e-n+2}$, $f \to$ no. of regions

Note— In any simple, connected, planar graph with f regions, n vertices and e edges $(e > 2)$ the following inequalities must hold:

$e \geq \frac{3}{2}f$ or $\boxed{3f \leq 2e}$

and $\boxed{e \leq 3n-6}$

This is necc but not a ~~suff.~~ cond^n for the planarity of a graph.

e.g. in K_5 → complete graph with 5 vertices.

$n = 5$, $e = \frac{n(n-1)}{2} = \frac{5 \times 4}{2} = 10$

$3n-6 = 3 \times 5 - 6 = 9$

$e = 10 \nleq 3n-6 = 9$

$\therefore K_5$ is not planar

$K_{3,3}$ → $e = 9$, $3n-6 = 3 \times 6 - 6 = 12$

$e \leq 3n-6$

but $K_{3,3}$ is non-planar

Note— A complete bipartite graph $K_{m,n}$ is planar if m or n is less than or equal to 2.

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